Strong Maximum Principle

Brian Krummel

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We will continue from the Weak Maximum Principle lecture(s) to consider the *strong maximum principle*, which states that a subsolution to an elliptic differential equation on a bounded domain Ω only attains its maximum value on the boundary of Ω unless the subsolution is a constant function.

As before, we will consider

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu \geq 0 \text{ in } \Omega,$$

where $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ and a^{ij}, b^i, and c are (real-valued) functions on Ω. We will assume the strong condition on L of uniform ellipticity, i.e.

$$\lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2 \text{ for all } x \in \Omega,\ \xi \in \mathbb{R}^n$$

for some $\lambda(x)$ and $\Lambda(x)$ such that $0 < \lambda(x) \leq \Lambda(x)$ and

$$\sup_{x \in \Omega} \frac{\Lambda(x)}{\lambda(x)} < \infty.$$

Note that we can assume that $\lambda(x) = 1$ for all $x \in \Omega$ by replacing L with $\lambda^{-1} L$, in which case uniform ellipticity is equivalent to $\sup_{x \in \Omega} \Lambda(x) < \infty$. Recall from the weak maximum principle lectures that when considering maximum principles, we have three cases depending on the sign of c to determine what type of maximum values $u(y)$ of u for $y \in \overline{\Omega}$ that we consider:

(a) When $c = 0$ on Ω, we consider the maximum value of u.

(b) When $c \leq 0$ on Ω, we consider nonnegative maximum values of u, i.e. maximum values where $u(y) \geq 0$.

(c) When we assume no sign restriction on c, we consider zero maximum values of u, i.e. maximum values where $u(y) = 0$.

Lemma 1 (Hopf boundary point lemma)**.** *Let Ω be an open set in $\mathbb{R}^n$ and $y \in \partial\Omega$. Suppose $u : \Omega \cup \{y\} \to \mathbb{R}$ such that $u \in C^2(\Omega)$ and*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu \geq 0 \text{ in } \Omega$$

for some functions a^{ij}, b^i, and c on Ω. Suppose L is a uniformly elliptic operator and

$$\sup_\Omega \frac{|b^i|}{\lambda} + \sup_\Omega \frac{|c|}{\lambda} < \infty.$$

Suppose

(i) u *is continuous at* y,

(ii) $u(y) > u(x)$ *for all* $x \in \Omega$,

(iii) Ω *satisfies the interior sphere condition at* y, *i.e. there is a ball* $B = B_R(z) \subset \Omega$ *with* $y \in \partial B$, *and*

(iv) one of the following holds true:

(a) $c = 0$ *on* Ω,

(b) $c \leq 0$ *on* Ω *and* $u(y) \geq 0$.

(c) c *has any sign and* $u(y) = 0$.

Let ν *be the outward unit normal to* $B_R(z)$ *at* y. *Then, if* $(\partial u/\partial \nu)(y)$ *exists,*

$$\frac{\partial u}{\partial \nu}(y) > 0.$$

Proof of Cases (a) and (b). Let

$$\beta = \sup_{\Omega} \frac{|\beta|}{\lambda} + \sup_{\Omega} \frac{|c|}{\lambda}.$$

Let A be the annulus $A = B_R(y) \setminus \overline{B_{R/2}(y)}$ for $\rho \in (0, R)$ to be determined,

$$v(x) = e^{-\alpha|x-z|^2} - e^{-\alpha R^2} \text{ for } x \in A$$

for some constant $\alpha > 0$ to be determined. We want to compare $u(x)$ to $u(y) - \varepsilon v(x)$ for $\varepsilon > 0$. For $x \in A$,

$$\begin{aligned} Lv(x) &= e^{-\alpha|x-z|^2}\left(\sum_{i,j=1}^{n} 4\alpha^2 a^{ij}(x_i - z_i)(x_j - z_j) - \sum_{i=1}^{n} 2\alpha(a^{ii} + b^i(x_i - z_i))\right) + c(e^{-\alpha|x-z|^2} - e^{-\alpha R^2}) \\ &\geq \lambda e^{-\alpha|x-z|^2}\left(4\alpha^2(R/2)^2 - 2\alpha(n\Lambda/\lambda + \beta R) - \beta\right) \\ &> 0 \end{aligned}$$

provided α is chosen sufficiently large depending on R, Λ, and β. By linearity and (iv),

$$L(u(y) - \varepsilon v) = cu(y) - \varepsilon Lv < 0 \text{ in } A$$

for all $\varepsilon > 0$. Hence $Lu > L(u(y) - \varepsilon v)$ in A. By (ii) ($u(y) > u(x)$ for all $x \in \Omega$), $u \leq u(y) = u(y) - \varepsilon v$ on $\partial B_R(z)$ and $u < u(y) - \varepsilon v$ on $\partial B_{R/2}(z)$ provided $\varepsilon > 0$ is sufficiently small. By the comparison principle,

$$u \leq u(y) - \varepsilon v(x) \text{ for all } x \in \overline{A}.$$

In other words $u(x) - u(y) + \varepsilon v(x)$ is a nonpositive function on $\overline{A}$ attaining a maximum value of zero at $x = y$, so

$$\frac{\partial(u - u(y) + \varepsilon v)}{\partial \nu}(y) = \frac{\partial u}{\partial \nu}(y) + \varepsilon \frac{\partial v}{\partial \nu}(y) \geq 0,$$

i.e.

$$\frac{\partial u}{\partial \nu}(y) \geq -\varepsilon \frac{\partial v}{\partial \nu}(y) = 2\varepsilon\alpha R e^{-\alpha R^2} > 0.$$

□

Proof of Case (c). Exercise. Let $L_0 = a^{ij}D_{ij} + b^iD_i - c_-$, where $c = c_+ - c_-$ for $c_+ = \max\{c, 0\}$ and $c_- = \max\{-c, 0\}$. Since

$$L_0u = Lu - c_+u \geq 0 \text{ in } \Omega,$$

using the fact that $u(x) < u(y) = 0$ for all $x \in \Omega$. By Case (b),

$$\frac{\partial u}{\partial \nu}(y) > 0.$$

□

Before moving on, note that if Ω is a C^2 domain, then Ω automatically satisfies the interior sphere condition.

Lemma 2. *Suppose Ω is a C^2 domain in $\mathbb{R}^n$. Then Ω satisfies the interior sphere condition at every $y \in \partial\Omega$.*

Proof. Suppose $y \in \partial\Omega$ and after translation suppose $y = 0$. Write $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ as $x = (x', x_n)$ where $x' = (x_1, x_2, \ldots, x_{n-1})$. Since Ω is a C^2 domain, after a rotation we may write

$$\Omega \cap B_\rho(0) = \{(x', x_n) \in B_\rho(0) : x_n > g(x')\}$$

for some $\rho > 0$ and some C^2 function $g : B^{n-1}_\rho(0) \to \mathbb{R}$ such that $g(0) = 0$ and $Dg(0) = 0$. Note that by Taylor's theorem,

$$|g(x')| \leq M|x'|^2 \tag{1}$$

for all $x' \in B_\rho(0)$ for some constant $M \in (0, \infty)$. We claim that for some $R \in (0, \rho/2)$, the open ball $B_R(Re_n)$ is contained in Ω, where $e_1, e_2, \ldots, e_n$ are the standard basis for $\mathbb{R}^n$. Note that $0 \in \partial B_R(Re_n)$. Suppose $x = (x', x_n) \in B_R(Re_n) \setminus \Omega$. Since $x \in B_R(Re_n)$,

$$|x'|^2 + (x_n - R)^2 < R^2;$$

that is

$$|x'|^2 + x_n^2 - 2Rx_n < 0. \tag{2}$$

By (1) and (2),

$$x_n < g(x') \leq M|x'|^2 < M(2Rx_n - x_n^2) \leq 2MRx_n.$$

Thus if we choose $R < 1/(2M)$, then $B_R(Re_n) \subset \Omega$. □

Theorem 1 (Strong maximum principle)**.** *Let Ω be a domain set (i.e. connected open set) in $\mathbb{R}^n$. Suppose $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies*

$$Lu = a^{ij}D_{ij}u + b^iD_iu + cu \geq 0 \text{ in } \Omega$$

for some functions a^{ij}, b^i, and c on Ω. Suppose L is a uniformly elliptic operator and

$$\sup_\Omega \frac{|b^i|}{\lambda} + \sup_\Omega \frac{|c|}{\lambda} < \infty.$$

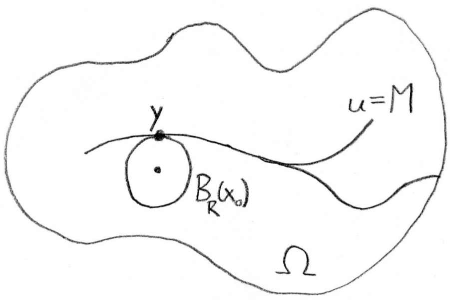
Then:

(a) If $c = 0$ on Ω, u cannot acheive its maximum value in the interior of Ω unless u is constant.

(b) If $c \leq 0$ on Ω, u cannot acheive a non-negative maximum in the interior of Ω unless u is constant on Ω.

(c) Regardless of the sign of c, u cannot acheive a maximum value of zero in the interior of Ω unless $u \equiv 0$.

Proof. Suppose u is non-constant and u acheives its maximum value M in the interior of Ω. Let $\Omega_M = \{x \in \Omega : u(x) < M\}$. Let x_0 be a point in Ω_M that is closer to $\partial\Omega_M$ than to $\partial\Omega$. Consider the largest open ball $B = B_R(x_0)$ centered at x_0 that is contained in Ω_M. Let y be the point where ∂B touches $\Omega \setminus \Omega_M$. Then $y \in \partial B$, $u(y) = M$, and $u < M$ in ∂B, so by the Hopf boundary point



lemma, $Du(y) \neq 0$. But on the other hand y is a maximum of u in the interior Ω, so $Du(y) = 0$. Thus we reach a contradiction. □

Corollary 1. *Let Ω be an open set in $\mathbb{R}^n$. Let*

$$L = a^{ij} D_{ij} + b^i D_i + c$$

for some functions a^{ij}, b^i, and c on Ω and suppose L is a uniformly elliptic operator and

$$\sup_{\Omega} \frac{|b^i|}{\lambda} + \sup_{\Omega} \frac{|c|}{\lambda} < \infty.$$

Suppose $u, v \in C^2(\Omega)$ such that $u \leq v$ on Ω and $Lu \geq Lv$ on Ω. Then either $u \equiv v$ on Ω or $u < v$ on Ω.

Proof. Exercise. □

Corollary 2. *Let Ω be a C^2 domain in $\mathbb{R}^n$ and $y \in \partial\Omega$. Let*

$$L = a^{ij} D_{ij} + b^i D_i + c$$

for some functions a^{ij}, b^i, and c on Ω and suppose L is a uniformly elliptic operator and

$$\sup_{\Omega} \frac{|b^i|}{\lambda} + \sup_{\Omega} \frac{|c|}{\lambda} < \infty.$$

Suppose $u, v \in C^1(\Omega \cup \{y\}) \cap C^2(\Omega)$ such that $u \leq v$ on Ω, $Lu \geq Lv$ on Ω, $u(y) = v(y)$, and $Du(y) = Dv(y)$. Then $u \equiv v$ on Ω.

Proof. Exercise. □

References: Gilbarg and Trudinger, Section 3.2.